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On the construction of covariant differential calculi on quantum homogeneous spaces

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Dedicated to the memory of Stanisłav Zakrzewski (1951–1998)

Abstract

Let $\mathcal{A}$ be a coquasitriangular Hopf algebra and $\mathcal{X}$ the subalgebra of $\mathcal{A}$ generated by a row of a matrix corepresentation $\mathbf{u}$ or by a row of $\mathbf{u}$ and a row of the contragredient corepresentation $\mathbf{u}^c$. In the paper left-covariant first order differential calculi on the quantum group $\mathcal{A}$ are constructed and the corresponding induced calculi on the left quantum space $\mathcal{X}$ are described. The main tool for these constructions are the L-functionals associated with $\mathbf{u}$. The results are applied to the quantum homogeneous space $GL_q(N)/GL_q(N-1)$. © 1999 Elsevier Science B.V. All rights reserved.

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0. Introduction

Based on the pioneering work of Woronowicz [19], a beautiful theory of bicovariant differential theory on quantum groups has been developed till now. A thorough treatment of this theory can be found in Chapter 14 of the monograph [6]. The theory of covariant differential calculi on quantum spaces, in contrast, is still at the very beginning and neither general methods for the construction of such calculi nor remarkable general results are known. Covariant differential calculi have been constructed and studied so far only on a few simple quantum spaces [1,3,9–12,16,17].

In this paper we are concerned with the construction of first order differential calculi (FODC) on subalgebras of a coquasitriangular Hopf algebra $\mathcal{A}$ which are generated by

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a row of a fixed corepresentation $\mathbf{u}$ or by a row of $\mathbf{u}$ and a row of the contragredient corepresentation $\mathbf{u}^c$ of $\mathcal{A}$. Such a subalgebra is a left quantum space of $\mathcal{A}$ with left coaction given by the restriction of the comultiplication. Our method of construction is easy to explain: The FODC on the quantum spaces are induced from appropriate *left-covariant* differential calculi on the quantum group $\mathcal{A}$. The main technical tool for the construction of the left-covariant calculi on $\mathcal{A}$ are the L-functionals associated with the corepresentation $\mathbf{u}$. We always try to be as simple and close to the classical situation as possible. Our approach has two important advantages: First, because of the close relationship between the calculi on the quantum space and on the quantum group the theory of L-functionals and other Hopf algebra techniques can be applied to the study of the calculi on the quantum space. Secondly, the simplicity of the constructed left-covariant calculi, in contrast to the usual bicovariant calculi, might be useful for doing explicit computations. Our guiding example are the quantum spheres associated with the quantum group $GL_q(N)$ (see [8,15] or [6, 11.6]). For these quantum spheres a classification of covariant differential calculi has been recently given by Welk [16]. As an application of our method we describe some of the main calculi occuring there as induced from left-covariant calculi on $GL_q(N)$. Strictly speaking, we derive the left-covariant counter-parts of these calculi, because in [16] right quantum spheres and right-covariant calculi are investigated.

This paper is organized as follows. Section 1 contains some preliminaries and collects some notation. In Sections 2 and 3 first order calculi on the left quantum spaces generated by a single row of $\mathbf{u}$ and $\mathbf{u}^c$, respectively, are investigated. Section 4 deals with the quantum space generated by a row of $\mathbf{u}$ and a row of $\mathbf{u}^c$. Four families of covariant FODC are constructed and the commutation rules between generators and their differentials are explicitly described. The application of the results to the fundamental corepresentations of the quantum groups $GL_q(N)$ and $SL_q(N)$ are discussed in Section 5. In Section 6 another interesting FODC on the quantum sphere is obtained from a particular *bicovariant* (!) calculus on $GL_q(N)$. The left-covariant differential calculi on the quantum groups have been so far only auxilary tools for the study of the induced FODC on the quantum spaces. In Section 7 the same idea is used in order to construct "reasonable" left-covariant FODC on the quantum groups $GL_q(N)$, $SL_q(N)$, $O_q(N)$ and $Sp_q(N)$ which are in many aspects close to the ordinary differential calculus on the corresponding Lie groups. In particular, the dimensions of these calculi coincide with the classical group dimensions.

1. Preliminaries

Throughout this paper $\mathcal{A}$ is a coquasitriangular complex Hopf algebra and $\mathbf{r}$ denotes a fixed universal r-form of $\mathcal{A}$ (see, for instance, [7] or [6], Section 10.1], for these notions). The comultiplication, the counit and the antipode of $\mathcal{A}$ are denoted by Δ, ε and S, respectively. We shall use the Sweedler notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$ for the comultiplication of $\mathcal{A}$. Let us recall that a Hopf algebra $\mathcal{A}$ is called *coquasitriangular* if it is equipped with a linear functional $\mathbf{r}$ on $\mathcal{A} \otimes \mathcal{A}$ which is invertible with respect to the convolution multiplication and satisfies the following conditions for arbitrary elements $a, b, c \in \mathcal{A}$:

$$\mathbf{r}(ab \otimes c) = \mathbf{r}(a \otimes c_{(1)})\mathbf{r}(b \otimes c_{(2)}),$$

$$\mathbf{r}(a \otimes bc) = \mathbf{r}(a_{(1)} \otimes c)\mathbf{r}(a_{(2)} \otimes b), \tag{1}$$

$$\mathbf{r}(a_{(1)} \otimes b_{(1)})a_{(2)}b_{(2)} = \mathbf{r}(a_{(2)} \otimes b_{(2)})b_{(1)}a_{(1)}. \tag{2}$$

Such a linear form $\mathbf{r}$ is called a *universal r-form* of the Hopf algebra $\mathcal{A}$. The convolution inverse of $\mathbf{r}$ is denoted by $\bar{\mathbf{r}}$. We shall write $\mathbf{r}(a, b) := \mathbf{r}(a \otimes b), a, b \in \mathcal{A}$.

Further, $\mathbf{u} = (u^i_j)_{i,j=1,\dots,n}$ denotes a fixed n-dimensional *matrix corepresentation* of $\mathcal{A}$, that is, $\mathbf{u}$ is an $n \times n$-matrix of elements u^i_j of $\mathcal{A}$ such that

$$\Delta(u^i_j) = \sum_{k=1}^{n} u^i_k \otimes u^k_j \quad \text{and} \quad \varepsilon(u^i_j) = \delta_{ij} \quad \text{for } i, j, = 1, \dots, n.$$

We define the L-functionals $l^{\pm i}_j$ and the R-matrix $\hat{R}$ associated with the corepresentation $\mathbf{u}$ by

$$l^{+i}_j(\cdot) = \mathbf{r}(\cdot \otimes u^i_j), \quad l^{-i}_j(\cdot), = \bar{\mathbf{r}}(u^i_j \otimes \cdot), \quad \hat{R}^{ji}_{nm} := \mathbf{r}(u^i_n, u^j_m).$$

The Hopf dual of the Hopf algebra $\mathcal{A}$ is denoted by $\mathcal{A}^\circ$. The L-functionals $l^{\pm i}_j$ belong to $\mathcal{A}^\circ$. From (1) it follows that

$$\Delta(l^{\pm i}_j) = \sum_{k=1}^{n} l^{\pm k} \otimes l^{\pm k}_j, \quad i, j, = 1, \dots, n.$$

These and the following relations will be often used in this paper:

$$(l^{+i}_j, u^k_l) = \hat{R}^{ik}_{lj}, \qquad (l^{-i}_j, u^k_l) = (\hat{R}^{-1})^{ik}_{lj} = \bar{\mathbf{r}}(u^i_j, u^k_j),$$
$$(S(l^{+i}_j), u^k_l)) = (\hat{R}^{-1})^{ki}_{jl}, \qquad (S(l^{-i}_j), u^k_l)) = \hat{R}^{ki}_{jl}.$$

Formula (2) implies that the matrix $\hat{R}$ and hence also $\hat{R}^{-1}$ intertwine the tensor product corepresentation $\mathbf{u} \otimes \mathbf{u}$.

Suppose that $\mathcal{X}$ is a subalgebra $\mathcal{X}$ of $\mathcal{A}$ such that $\Delta(\mathcal{X}) \subseteq \mathcal{A} \otimes \mathcal{X}$. Then $\mathcal{X}$ is a left $\mathcal{X}$-comdodule algebra or equivalently a *left quantum space* of $\mathcal{A}$ with left coaction φ given by the restriction $\Delta\lceil\mathcal{X}$ of the comultiplication of $\mathcal{A}$. As in [6], such a subalgebra $\mathcal{X}$ will be called a *left quantum homogenous space* of the Hopf algebra $\mathcal{A}$.

A *first order differential calculus* (FODC) over $\mathcal{X}$ is an $\mathcal{X}$-bimodule Γ equipped with a linear mapping d: $\mathcal{X} \to \Gamma$, called the differentiation, such that:

(i) d satisfies the Leibniz rule $\mathrm{d}(xy) = x\,\mathrm{d}y + \mathrm{d}x\,y$ for any $x, y \in \mathcal{X}$,

(ii) Γ is the linear span of elements $x\,\mathrm{d}y\,z$ with $x, y, z \in \mathcal{X}$.

An FODC Γ over $\mathcal{X}$ is called *left-covariant* if there exists a linear mapping $\Phi : \Gamma \to \mathcal{X} \otimes \Gamma$ such that $\Phi(x\,\mathrm{d}y) = \Delta(x)(\mathrm{id} \otimes \mathrm{d})\Delta(y)$ for all $x, y \in \mathcal{X}$. For a left-covariant FODC Γ of $\mathcal{X}$ the elements of the vector space ${}_{\mathrm{inv}}\Gamma = \{\eta \in \Gamma | \Phi(\eta) = 1 \otimes \eta\}$ are called *left-invariant* one-forms. A left-covariant FODC Γ of $\mathcal{X}$ is called *inner* if there exists a left-invariant one-form $\theta \in {}_{\mathrm{inv}}\Gamma$ such that

$$\mathrm{d}x = \theta x - x\theta, \quad x \in \mathcal{X}.$$

Let Γ be a left-covariant FODC on the Hopf algebra $\mathcal{A}$ itself such that $\dim \Gamma := \dim {}_{\mathrm{inv}}\Gamma$ is finite-dimensional. We briefly recall a few facts from the general theory of these calculi (see [2,19] or [6], Section 14.1], for more details) that will be used in what follows. Such an FODC Γ is characterized by a finite-dimensional subspace $\mathcal{T}$ of $\mathcal{A}^\circ$, called the *quantum tangent space* of Γ, and there is a canonical non-generate bilinear form $(\cdot,\cdot)$ on $\mathcal{T} \times {}_{\mathrm{inv}}\Gamma$. If $\{X_i; i \in I\}$ and $\{\theta_i; i \in I\}$ are dual bases of $\mathcal{T}$ and ${}_{\mathrm{inv}}\Gamma$ with respect to this bilinear form, then the differentiation d of the FODC Γ can be expressed by

$$\mathrm{d}a = \sum_i a_{(1)} X_i(a_{(2)})\theta_i, \quad a \in \mathcal{A}. \tag{3}$$

The commutation relations between the elements of $\mathcal{A}$ and left-invariant one-forms of Γ are given by

$$\theta_i a = \sum_k a_{(1)} f^i_k(a_{(2)})\theta_k, \quad a \in \mathcal{A}, \tag{4}$$

where f^i_k are the functionals on $\mathcal{A}$ are determined by the equation

$$\Delta(X_k) - \varepsilon \otimes X_k = \sum_i X_i \otimes f^i_k. \tag{5}$$

Let $\omega : \mathcal{A} \to {}_{\mathrm{inv}}\Gamma$ be the canonical projection defined by $\omega(a) = S(a_{(1)})\,\mathrm{d}a_{(2)}$ for $a \in \mathcal{A}$. Then one has

$$(X, \omega(a)) = (X, a) \quad \text{for } X \in \mathcal{T} \text{ and } a \in \mathcal{A}. \tag{6}$$

If Γ is an FODC of $\mathcal{A}$ with differentiation d, then $\tilde{\Gamma} := \mathcal{X}\,\mathrm{d}\mathcal{X}\,\mathcal{X}$ is obviously an FODC of the subalgebra $\mathcal{X}$ with differentiation $d\lceil\mathcal{X}$. We call $\tilde{\Gamma}$ the *induced* FODC of the FODC Γ of $\mathcal{A}$. Clearly, if Γ is left-covariant on the quantum group $\mathcal{A}$, then so is $\tilde{\Gamma}$ on the left quantum space $\mathcal{X}$.

Our constructions of left-covariant FODC on $\mathcal{A}$ are based on the following lemma.

Lemma 1. *A finite-dimensional vector space $\mathcal{T}$ of $\mathcal{A}^\circ$ is the quantum tangent space of a left-covariant FODC of $\mathcal{A}$ if and only if $X(1) = 0$ and $\Delta(X) - \varepsilon \otimes X \in \mathcal{X} \otimes \mathcal{A}^\circ$ for all $X \in \mathcal{X}$.*

Proof. [13, Lemma 1], or [6, Proposition 14.5]. □

2. Quantum spaces generated by a row of u

Let $\mathcal{X}$ denote the unital subalgebra of $\mathcal{A}$ generated by the entries of the last row of the matrix $\mathbf{u}$, that is, by the elements $x_i := u^i_n$, $i = 1, \ldots, n$. Clearly, $\mathcal{X}$ is a left quantum homogeneous space of $\mathcal{A}$ with left coaction $\varphi = \Delta\lceil\mathcal{X}$ determined by

$$\varphi(x_i) \equiv \Delta(u^i_n) = \sum_{j=1}^n u^i_j \otimes x_j, \quad i = 1, \ldots, n. \tag{7}$$

In this section we shall construct an n-dimensional left-covariant FODC Γ on the Hopf algebra $\mathcal{A}$ which induces an FODC $\Gamma^{\mathcal{X}}$ on $\mathcal{X}$ such that the differentials $\mathrm{d}x_1, \dots, \mathrm{d}x_n$ form a free left $\mathcal{X}$-module basis of $\Gamma^{\mathcal{X}}$.

First we define an FODC Γ of $\mathcal{A}$. Let $\mathcal{T}^{\mathcal{X}}$ be the linear span of functionals

$$X_i := \alpha^{-1} l_i^{-n} l_n^{-n}, \quad i = 1, \dots, n-1, \quad \text{and} \quad X_n := \alpha^{-1}((l_n^{-n})^2 - \varepsilon)$$

on $\mathcal{A}$, where α is non-zero complex number that will be specified by formula (11) below. We assume that

$$l_n^{-m} = 0 \quad \text{if } m < n. \tag{8}$$

Since $\Delta(l_n^{-n}) = \sum_i l_i^{-n} \otimes l_n^{-i}$, this assumption implies in particular that $\Delta(l_n^{-n}) = l_n^{-n} \otimes l_n^{-n}$, so that l_n^{-n} is a character of the algebra $\mathcal{A}$ (that is, $l_n^{-n}(ab) = l_n^{-n}(a) l_n^{-n}(b)$ for $a, b \in \mathcal{A}$ and $l_n^{-n}(1) = 1$). Using the relation $\Delta(l_n^{-n}) = l_n^{-n} \otimes l_n^{-n}$ we get

$$\Delta(X_i) - \varepsilon \otimes X_i = \sum_{j=1}^{n} X_j \otimes l_i^{-j} l_n^{-n}, \quad i = 1, \dots, n-1,$$

$$\Delta(X_n) - \varepsilon \otimes X_n = X_n \otimes (l_n^{-n})^2.$$

Because of (8), the latter equations can be written in the compact form

$$\Delta(X_i) - \varepsilon \otimes X_i = \sum_{j=1}^{n} X_i \otimes l_i^{-j} l_n^{-n}, \quad i = 1, \dots, n. \tag{9}$$

Since obviously $X(1) = 0$ and $\Delta(X) - \varepsilon \otimes X \in \mathcal{T}^{\mathcal{X}} \otimes \mathcal{A}^\circ$ for all $X \in \mathcal{T}^{\mathcal{X}}$ by (9), it follows from Lemma 1 that there exists a left-covariant FODC Γ on $\mathcal{A}$ such that $\mathcal{T}^{\mathcal{X}}$ is the quantum tangent space of Γ.

Let us suppose in addition that

$$(l_i^{-n}, u_n^j) = 0 \quad \text{if } i \neq j, \ i, j = 1, \dots, n, \tag{10}$$

$$\alpha := (l_i^{-n} l_n^{-n}, u_n^i) = ((l_n^{-n})^2, u_n^n) - 1 \neq 0 \quad \text{for } i = 1, \dots, n-1. \tag{11}$$

We abbreviate $c_- := (l_n^{-n}, u_n^n)$. Then we have $\alpha = c_-^2 - 1$.

For $i = 1, \dots, n$, let θ_i denote the left-invariant one-form $\omega(u_n^i) \equiv \sum_k S(u_k^i)\, \mathrm{d}u_n^k$ of Γ. The assumptions (10) and (11) imply that $(X_j, u_n^i) = \delta_{ij}$ and so by formula (6) that

$$(X_j, \theta_i) = (X_j, \omega(u_n^k)) = (X_j, u_n^i) = \delta_{ij} \tag{12}$$

for $i, j = 1, \dots, n$. In particular we conclude that the functionals $X_i, \dots, X_n$ are linearly independent, so that the FODC Γ is n-dimensional. Further, (12) shows that $\{\theta_1, \dots, \theta_n\}$ and $\{X_1, \dots, X_n\}$ are dual bases of ${}_{\mathrm{inv}}\Gamma$ and $\mathcal{T}^{\mathcal{X}}$, respectively. Therefore, comparing (5) and (9) and using (4), (7) and (8), we obtain for $r, j = 1, \dots, n$,

$$\theta_r x_j = \sum_{k,s} u_k^j (l_s^{-r} l_n^{-n}, u_n^k) \theta_s = \sum_{k,m,s} u_k^j (l_s^{-r}, u_m^k)(l_n^{-n}, u_n^m) \theta_s$$

$$= \sum_{k,s} c^{-1} (\hat{R}^{-1})_{ns}^{rk} u_k^j \theta_s. \tag{13}$$

These relations lead to the following commutation rule between the one-forms θ_r and elements of the algebra $\mathcal{X}$:

$$\theta_r x = \sum_{s=1}^{n} x_{(1)} \bar{\mathbf{r}}(u^r_s, x_{(2)})(l^{-n}_n, x_{(3)})\theta_s, \quad x \in \mathcal{X}. \tag{14}$$

Indeed, if x is the generator x_j of $\mathcal{X}$, then the third expression of (13) can be rewritten as the right-hand side of (14). Using the facts that l^{-n}_n is a character and that $\bar{\mathbf{r}}_{21}$ is also a universal r-form of $\mathcal{A}$ (see [6, Proposition 10.2(iv)]), one easily verifies that (14) holds for a product $x'x''$ provided that it holds for both factors x' and x''. Thus, (14) is valid for arbitrary elements x of $\mathcal{X}$.

Next we turn to the FODC $\Gamma^{\mathcal{X}}$ of $\mathcal{X}$.

Proposition 2.

(i) *The FODC Γ of $\mathcal{A}$ induces a left-covariant FODC $\Gamma^{\mathcal{X}}$ of $\mathcal{X}$ such that the set* $\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\}$ *is a free left $\mathcal{X}$-module basis of $\Gamma^{\mathcal{X}}$. The $\mathcal{X}$-bimodule structure of $\Gamma^{\mathcal{X}}$ is determined by the commutation relations*

$$\mathrm{d}x_i x_j = (l^{-n}_n, u^n_n) \sum_{k,m=1}^{n} (\hat{R}^{-1})^{ij}_{km} x_k \, \mathrm{d}x_m, \quad i, j = 1, \ldots, n, \tag{15}$$

or equivalently by

$$\mathrm{d}x_i x = \sum_{m=1}^{n} \mathbf{r}(u^i_m, x_{(1)}) x_{(2)} (l^{-n}_n, x_{(3)}) \, \mathrm{d}x_m, \quad x \in \mathcal{X}. \tag{16}$$

(ii) *For the differentiation d of the FODC $\Gamma^{\mathcal{X}}$ of $\mathcal{X}$ we have*

$$\mathrm{d}x = \alpha^{-1}(\theta_n x - x\theta_n), \quad x \in \mathcal{X}, \tag{17}$$

Proof.

(i) First we prove formula (15). Since $(X_r, u^k_n) = \delta_{kr}$, it follows from (3) that

$$\mathrm{d}x_i \equiv \mathrm{d}u^i_n = \sum_{k,r} u^i_k X_r(u^k_n)\theta_r = \sum_r u^i_r \theta_r. \tag{18}$$

Using (13), (18) and the fact that $\hat{R}^{-1}$ intertwines the tensor product corepresentation $\mathbf{u} \otimes \mathbf{u}$, we obtain

$$\begin{aligned} \mathrm{d}x_i x_j &= \sum_k u^i_k \theta_k u^j_n = \sum_{k,m,s} c_- u^i_k u^j_m (\hat{R}^{-1})^{km}_{ns} \theta_s \\ &= \sum_{k,m,s} c_- (\hat{R}^{-1})^{ij}_{km} u^k_n u^m_s \theta_s = \sum_{k,m} c_- (\hat{R}^{-1})^{ij}_{km} x_k \, \mathrm{d}x_m; \end{aligned}$$

which proves (15). Formula (16) can be derived from (15) similarly as (14) was from (13). From (15) combined with the Leibniz rule it follows that $\Gamma^{\mathcal{X}} \equiv \mathcal{X}\, \mathrm{d}\mathcal{X}\mathcal{X}$ is equal to Lin $\{x\, \mathrm{d}x_i;\, x \in \mathcal{X}, i = 1, \ldots, n\}$. Suppose that $\sum_i a_i \, \mathrm{d}x_i = 0$ for certain elements

$a_i \in \mathcal{X}$. Then we have $\sum_{i,k} a_i u^i_k \theta_k = 0$. Since $\{\theta_1, \ldots, \theta_n\}$ is a free left $\mathcal{A}$-module basis of Γ, the latter yields $\sum_i a_i u^i_k = 0$ for $k = 1, \ldots, n$ and hence $\sum_{i,k} a_i u^i_k S(u^k_j) = a_j = 0$ for all $j = 1, \ldots, n$. Thus, $\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\}$ is a free left $\mathcal{X}$-module basis of $\Gamma^{\mathcal{X}}$.

(ii) By (10) and (11) we have $c_-(\hat{R}^{-1})^{nk}_{ns} = (l^{-n}_s, u^k_n)(l^{-n}_n, u^n_n) = \delta_{ks}(l^{-n}_s l^{-n}_n, u^k_n) = \delta_{ks}\alpha$ and $c_-(\hat{R}^{-1})^{nk}_{nn} = c_-(l^{-n}_n, u^k_n) = \delta_{kn} c^2_-$ for $s = 1, \ldots, n-1$ and $k = 1, \ldots, n$. Inserting this into (13) using (18) we obtain

$$\begin{aligned} \theta_n x_j &= \sum_{k=1}^{n-1} \alpha u^j_k \theta_k + c^2_- u^j_n \theta_n \\ &= \sum_{k=1}^{n} \alpha u^j_k \theta_k + (c^2_- - \alpha) u^j_n \theta_n = \alpha\, \mathrm{d}x_j + x_j \theta_n, \end{aligned}$$

which proves (17) in the case $x = x_j$. Since both sides of (17), considered as mappings of $\mathcal{X}$ to $\Gamma^{\mathcal{X}}$, satisfy the Leibniz rule, (17) holds for all $x \in \mathcal{X}$. □

Remarks.

(1) Since the left-invariant form $\theta_n \in \Gamma$ does not belong to the $\mathcal{X}$-bimodule $\Gamma^{\mathcal{X}}$, formula (17) does not mean that the FODC $\Gamma^{\mathcal{X}}$ is inner. It expresses rather the differentiation d of $\Gamma^{\mathcal{X}}$ by means of an extended bimodule in the sense of Woronowicz (see [18]). But for the FODC $\Gamma^{\mathcal{Z}}_1$ of the larger algebra $\mathcal{Z}$ considered in Section 4 the form θ_n is in $\Gamma^{\mathcal{Z}}_1$ and makes $\Gamma^{\mathcal{Z}}_1$ into an inner FODC (see Proposition 4 (iii) below).

(2) If $\mathcal{A}$ is one of the coordinate Hopf algebras $\mathcal{O}(G_q)$, $G_q = GL_q(N), SL_q(N), O_q(N), Sp_q(N)$, then the conditions (8) and (19) below can be assumed without loss of generality. This follows from the particular form of the universal R-matrix for the corresponding Drinfeld–Jimbo algebras (see, for instance, [6, Theorem 8.17]).

3. Quantum spaces generated by a row of u^c

Let $\mathcal{Y}$ be the subalgebra of $\mathcal{A}$ generated by the elements $y_i := (\mathbf{u}^c)^i_n \equiv S(u^n_i)$, $i = 1, \ldots, n$, of the last row of the contragredient corepresentation $\mathbf{u}^c$. Then $\mathcal{Y}$ is a left quantum space of $\mathcal{A}$ with left coaction $\varphi = \Delta\lceil\mathcal{Y}$ given on the generators y_i by

$$\varphi(y_i) \equiv \Delta(S(u^n_i)) = \sum_{j=1}^{n} S(u^j_i) \otimes y_j, \quad i = 1, \ldots, n.$$

We shall proceed in a similar manner as in Section 2. But the considerations are technically slightly more complicated, because we have to deal with square and inverse of the antipode of $\mathcal{A}$.

Let β be a non-zero complex number and let $\mathcal{T}^{\mathcal{Y}}$ be the linear span of functionals

$$Y_i := \beta^{-1} S(l^{+i}_n) l^{-n}_n, \quad i = 1, \ldots, n-1,$$

and

$$Y_n := \beta^{-1}((l^{-n}_n)^2 - \varepsilon).$$

We assume that

$$l_m^{+n} = l_n^{-m} = 0 \text{ if } m < n \quad \text{and} \quad S(l_n^{\pm n}) = l_n^{\mp n}. \tag{19}$$

Similarly as in Section 2, we then get

$$\Delta(Y_i) - \varepsilon \otimes Y_i = \sum_{j=1}^{n} Y_j \otimes S(l_j^{+i}) l_n^{-n}, \quad i = 1, \ldots, n, \tag{20}$$

and $\mathcal{T}^{\mathcal{Y}}$ is the quantum tangent space of a left-covariant FODC Γ on $\mathcal{A}$.

Let us suppose in addition that there are numbers $\gamma_i \neq 0$, $i = 1, \ldots, n$, such that

$$S^2(u_j^i) = \gamma_i u_j^i \gamma_j^{-1}, \quad i, j = 1, \ldots, n, \tag{21}$$

and that

$$(l_n^{+i}, u_j^n) = 0 \quad \text{for } i \neq j, \;\; i, j = 1, \ldots, n, \tag{22}$$

$$\beta := (l_n^{+n} l_n^{+i}, u_i^n) = ((l_n^{+n})^2, u_n^n) - 1 \neq 0 \quad \text{for } i = 1, \ldots, n-1. \tag{23}$$

We set $c := (l_n^{+n}, u_n^n)$ and $\eta_i := \omega(S^{-1}(u_i^n)) = \sum_j u_i^j \, \mathrm{d}S^{-1}(u_j^n)$ for $i = 1, \ldots, n$. Since $S(l_n^{\pm n}) = l_n^{\mp n}$ by (19), we have $c_- = (l_n^{-n}, u_n^n) = c^{-1}$ and $\beta = c^2 - 1$. It is straightforward to check that (19), (22) and (23) imply that

$$(Y_j, \eta_i) = (Y_j, S^{-1}(u_i^n)) = \delta_{ij} \quad \text{for } i, j = 1, \ldots, n. \tag{24}$$

Therefore, the FODC $\Gamma^{\mathcal{Y}}$ is n-dimensional. From (20), (19) and (21) we get

$$\begin{aligned} \eta_r y_j &= \sum_{k,m,s} S(u_j^k)(S(l_r^{+s}),\, S(u_k^m))(l_n^{-n}, S(u_m^n))\eta_s \\ &= \sum_{k,s} c\gamma_n \gamma_k^{-1} \hat{R}_{ks}^{sn} S(u_j^k)\eta_s, \end{aligned} \tag{25}$$

for $j, r = 1, \ldots, n$. The first equality combined with the formulas $(S(l_r^{+s}), \cdot) = \mathbf{r}(S(\cdot), u_r^s) = \bar{\mathbf{r}}(\cdot, u_r^s)$ leads to the following form of the commutation relations:

$$\eta_r y = \sum_{s=1}^{n} y_{(1)} \bar{\mathbf{r}}(y_{(2)}, u_r^s)(l_n^{-n}, y_{(3)})\eta_s, \quad y \in \mathcal{Y}.$$

Let $\Gamma^{\mathcal{Y}} := \mathcal{Y}\, \mathrm{d}\mathcal{Y}\mathcal{Y}$ be the FODC on $\mathcal{Y}$ induced by the FODC Γ on $\mathcal{A}$.

Proposition 3.

(i) *$\Gamma^{\mathcal{Y}}$ is a left-covariant FODC on $\mathcal{Y}$ with the free left $\mathcal{Y}$-module basis* $\{\mathrm{d}y_1, \ldots, \mathrm{d}y_n\}$ *and with $\mathcal{Y}$-bimodule structure given by the relations*

$$\mathrm{d}y_i y_j = (l_n^{+n}, u_n^n) \sum_{k,m=1}^{n} \hat{R}_{ji}^{mk} y_k \, \mathrm{d}y_m, \quad i, j = 1, \ldots, n, \tag{26}$$

or equivalently by

$$\mathrm{d}y_i y = \sum_{m=1}^{n} \bar{\mathbf{r}}(y_{(1)}, u_i^m) y_{(2)} (l_n^{-n}, y_{(3)}) \,\mathrm{d}y_m, \quad y \in \mathcal{Y}. \tag{27}$$

(ii) *For any* $y \in \mathcal{Y}$ *we have* $\mathrm{d}y = \beta^{-1}(\eta_n y - y\eta_n)$.

Proof.

(i) It suffices to prove formula (26). First we note the (24) and (21) imply that

$$\begin{aligned} \mathrm{d}y_i &= \sum_{k,r} S(u_j^k)(Y_r, S(u_i^n))\eta_r = \sum_{k,r} \gamma_n \gamma_r^{-1} S(u_i^k)(Y_r, S^{-1}(u_k^n))\eta_r \\ &= \sum_r \gamma_n \gamma_r^{-1} S(u_i^r)\eta_r. \end{aligned} \tag{28}$$

If $\mathbf{r}$ is a universal r-form of $\mathcal{A}$, then so is $\bar{\mathbf{r}}_{21}$ and we have $\bar{\mathbf{r}}(a, S(b)) = \mathbf{r}(a, b)$ and $\mathbf{r}(S(a), b) = \bar{\mathbf{r}}(a, b)$, where $\bar{\mathbf{r}}_{21}(a, b) := \bar{\mathbf{r}}(b, a)$ and $a, b \in \mathcal{A}$ (see, for instance, [6]). Usign these facts and formulas (2) applied to $\bar{\mathbf{r}}_{21}$, (25), (22) and (21), we compute

$$\begin{aligned} \mathrm{d}y_i y_j &= \sum_r \gamma_n \gamma_r^{-1} S(u_i^r)\eta_r y_j \\ &= \sum_{k,m,r,s} \gamma_n \gamma_r^{-1} S(u_i^r) S(u_j^k)(l_n^{+n}, u_n^n)(l_r^{+s}, S^2(u_k^n)) u_s^m \,\mathrm{d}S^{-1}(u_m^n) \\ &= \sum_{k,m} c\gamma_n \gamma_r^{-1} S(u_i^r) \left(\sum_{k,s} S(u_j^k) u_s^m \bar{\mathbf{r}}_{21}(u_r^s, S(u_k^n)) \right) \mathrm{d}S^{-1}(u_m^n) \\ &= \sum_{k,m} c\gamma_n \gamma_i^{-1} S^{-1}(u_i^r) \left(\sum_{k,s} u_r^s S(u_k^n) \bar{\mathbf{r}}_{21}(u_s^m, S(u_j^k)) \right) \mathrm{d}S^{-1}(u_m^n) \\ &= \sum_{k,m} cS(u_k^n)\bar{\mathbf{r}}(S(u_j^k), \gamma_m u_i^m \gamma_i^{-1}) \,\mathrm{d}S^{-1}(\gamma_n u_m^n \gamma_m^{-1}) \\ &= \sum_{k,m} cS(u_k^n)\bar{\mathbf{r}}(S(u_j^k), S^2(u_i^m)) \,\mathrm{d}S(u_m^n) \\ &= \sum_{k,m} c\hat{R}_{ji}^{mk} y_k \,\mathrm{d}y_m. \end{aligned}$$

(ii) Since $c\hat{R}_{kn}^{sn} = (l_n^{+n}, u_n^n)(l_n^{+s}, u_n^n) = \delta_{ks}(l_n^{+n} l_n^{+k}, u_k^n) = \delta_{ks}\beta$ and $c\hat{R}_{kn}^{nn} = \delta_{kn}c^2$ by (22) and (23) for $s = 1, \ldots, n-1$ and $k = 1, \ldots, n$, it follows from (25) and (28) that

$$\begin{aligned} \eta_n y_j &= \sum_{k=1}^{n-1} \gamma_n \gamma_k^{-1} \beta S(u_j^k)\eta_k + c^2 S(u_j^n)\eta_n \\ &= \sum_{k=1}^{n} \gamma_n \gamma_k^{-1} \beta S(u_j^k)\eta_k + (c^2 - \beta) S(u_j^n)\eta_n = \beta \,\mathrm{d}y_j + y_j \eta_n \end{aligned}$$

which implies the assertion. □

4. Quantum spaces generated by a row of u and of u^c

Let $\mathcal{Z}$ denote the subalgebra of $\mathcal{A}$ generated by the elements $x_i = u^i_n$ and $y_i = S(u^n_i)$, $i = 1, \dots, n$. That is, $\mathcal{Z}$ is the subalgebra of $\mathcal{A}$ generated by the algebras $\mathcal{X}$ and $\mathcal{Y}$. Our aim in this section is to construct four classes $\Gamma^{\mathcal{Z}}_j$, $j = 1, 2, 3, 4$, of left-covariant FODC of $\mathcal{Z}$.

First let us fix some notations and assumptions which will be kept in force throughout the whole section. Let Z_n be a fixed group-like element of $\mathcal{A}^\circ$, that is, $Z_n(1) = 1$ and $\Delta(Z_n) = Z_n \otimes Z_n$. Then Z_n is invertible in $\mathcal{A}^\circ$ with inverse $Z_n^{-1} = S(Z_n)$. We retain the assumptions (10), (19), (21) and (22). In addition we suppose that

$$(S(l^{+i}_n), u^j_n) = (l^{-n}_i, S^{-1}(u^n_j)) = 0 \qquad \text{for } (i, j) \neq (n, n),\ \ i, j = 1, \dots, n, \tag{29}$$

$$\gamma := (l^{-n}_i l^{+n}_n, u^i_n) \neq 0 \quad \text{and} \quad \zeta := (l^{-n}_n l^{+i}_n, u^n_i) \neq 0$$
$$\text{are independent of } i = 1, \dots, n-1, \tag{30}$$

$$(l^{+n}_n, u^j_n) = (l^{-n}_n, u^n_j) = (Z_n, u^j_n) = (Z_n^{-1}, u^n_j) = 0 \quad \text{if } i \neq n, \tag{31}$$

$$\delta := (Z_n, u^n_n) \neq 1. \tag{32}$$

Clearly, we then have $\delta^{-1} = (Z_n^{-1}, u^n_n)$.

We now begin with the construction of the FODC $\Gamma^{\mathcal{Z}}_1$. Let $\mathcal{T}^{\mathcal{Z}}_1$ denote the linear span of functionals

$$X_i := \gamma^{-1}\delta^{-1} l^{-n}_i l^{+n}_n Z_n \ \text{ and } \ Y_i := \zeta^{-1}\delta S(l^{+i}_n) l^{+n}_n Z_n, \qquad i = 1, \dots, n-1,$$
$$X_n := (\delta - 1)^{-1}(Z_n - \varepsilon) \ \text{ and } \ Y_n := -\delta X_n = (\delta^{-1} - 1)^{-1}(Z_n - \varepsilon).$$

For $i = 1, \dots, n-1$, we have

$$\Delta(X_i) - \varepsilon \otimes X_i = \sum_{j=1}^{n-1} X_j \otimes l^{-j}_i l^{+n}_n Z_n + X_n \otimes (\delta - 1) X_i, \tag{33}$$

$$\Delta(Y_i) - \varepsilon \otimes Y_i = \sum_{j=1}^{n-1} Y_j \otimes S(l^{+i}_j) l^{-n}_n Z_n + Y_n \otimes (\delta^{-1} - 1) Y_i, \tag{34}$$

$$\Delta(X_n) - \varepsilon \otimes X_n = X_n \otimes Z_n, \quad \Delta(Y_n) - \varepsilon \otimes Y_n = Y_n \otimes Z_n. \tag{35}$$

Therefore, by Lemma 1, there exists a left-covariant FODC Γ_1 on $\mathcal{A}$ with quantum tangent space $\mathcal{T}^{\mathcal{Z}}_1$. As in Sections 2 and 3, we set

$$\theta_j := \omega(u^j_n) \ \text{ and } \ \eta_j := \omega(S^{-1}(u^n_j)), \qquad j = 1, \dots, n,$$

for the FODC Γ_1. From the assumptions (10), (22), (29), (31) and the definition of the functionals X_i, Y_i we immediately derive

$$(X_r, u^s_n) = (Y_r, S^{-1}(u^n_s)) = \delta_{rs}, \quad (X_i, S^{-1}(u^n_j)) = (Y_i, u^i_n) = 0 \tag{36}$$

and so

$$(X_r, \theta_s) = (Y_r, \eta_s) = \delta_{ks} \quad \text{and} \quad (X_i, \eta_j) = (Y_i, \theta_j) = 0$$

for all $i, j, r, s = 1, \dots, n$ such that $(i, j) \neq (n, n)$. That is, $\{\theta_1, \dots, \theta_n, \eta_1, \dots, \eta_{n-1}\}$ and $\{X_1, \dots, X_n, Y_1, \dots, Y_{n-1}\}$ and likewise $\{\theta_1, \dots, \theta_{n-1}, \eta_1, \dots, \eta_n\}$ and $\{X_1, \dots, X_{n-1},$

$Y_1, \dots, Y_n\}$ are dual bases of ${}_{\mathrm{inv}}\Gamma_1$ and $\mathcal{T}_1^{\mathcal{Z}}$, respectively. In particular, we see that the FODC Γ_1 has the dimension $\dim \mathcal{T}_1^{\mathcal{Z}} = 2n-1$. Moreover, the latter facts imply that formulas (18) and (28) hold for the differentiation d of the FODC Γ_1 as well. Further, from the formulas (4) and (33)–(35) we obtain the following commutation relations between the basis elements of ${}_{\mathrm{inv}}\Gamma_1$ and elements $a \in \mathcal{A}$:

$$\theta_r a = \sum_{s=1}^{n-1} a_{(1)}(l_s^{-r} l_n^{+n} Z_n, a_{(2)})\theta_s, \qquad \eta_s a = \sum_{s=1}^{n-1} a_{(1)}(S(l_r^{+s}) l_n^{+n} Z_n, a_{(2)})\eta_s, \tag{37}$$

$$\theta_n a = a_{(1)}(Z_n, a_{(2)})\theta_n + (\delta - 1)\sum_{s=1}^{n-1} a_{(1)}((X_s, a_{(2)})\theta_s + (Y_s, a_{(2)})\eta_s), \tag{38}$$

$$\eta_n a = a_{(1)}(Z_n, a_{(2)})\eta_n + (\delta^{-1} - 1)\sum_{s=1}^{n-1} a_{(1)}((X_s, a_{(2)})\theta_s + (Y_s, a_{(2)})\eta_s) \tag{39}$$

for $r = 1, \dots, n-1$.

Let $\Gamma_1^{\mathcal{Z}}$ denote the FODC of $\mathcal{Z}$ which induced by the FODC Γ_1 of $\mathcal{A}$.

Proposition 4.

(i) *For the $\mathcal{Z}$-bimodule $\Gamma_1^{\mathcal{Z}}$ we have the commutation relations*

$$\mathrm{d}x_i x_j = c\delta \sum_{k,m=1}^{n} (\hat{R}^{-1})^{ij}_{km} x_k \,\mathrm{d}x_m + (\delta - \gamma\delta - 1)(x_i \,\mathrm{d}x_j - x_i x_j \theta_n),$$

$$\mathrm{d}y_i y_j = c^{-1}\delta^{-1} \sum_{k,m=1}^{n} \hat{R}^{mk}_{ji} y_k \,\mathrm{d}y_m + (\delta^{-1} - \zeta\delta^{-1} - 1)(y_i \,\mathrm{d}y_j - y_i y_j \eta_n),$$

$$\mathrm{d}x_i y_j = c^{-1}\delta^{-1} \sum_{k,m=1}^{n} \hat{R}^{ki}_{mj} y_k \,\mathrm{d}x_m + (\delta - 1)(x_i \,\mathrm{d}y_j - x_i y_j \eta_n),$$

$$\mathrm{d}y_i x_j = c\delta \sum_{k,m=1}^{n} (\grave{R}^{-})^{ij}_{km} x_k \,\mathrm{d}y_m + (\delta^{-1} - 1)(y_i \,\mathrm{d}x_j - y_i x_j \theta_n),$$

where $(\grave{R}^{-})^{ij}_{km} := \bar{\mathbf{r}}(u^j_k, S^2(u^m_i))$, $i, j, k, m = 1, \dots, n$.

(ii) *The set* $\{\mathrm{d}x_1, \dots, \mathrm{d}x_n, \mathrm{d}y_1, \dots, \mathrm{d}y_n\}$ *generates* $\Gamma_1^{\mathcal{Z}}$ *as a left* $\mathcal{Z}$*-module. For arbitrary elements* $a_1, \dots, a_n, b_1, \dots, b_n \in \mathcal{Z}$, *the relation*

$$\sum_{i=1}^{n} (a_i \,\mathrm{d}x_i + b_i \,\mathrm{d}y_i) = 0 \tag{40}$$

is equivalent to the following set of equations:

$$a_j = \left(\sum_{i=1}^{n} a_i x_i\right) y_j, \quad b_j = \left(\sum_{i=1}^{n} b_i x_i\right) x_j \gamma_j \gamma_n^{-1} \qquad \textit{for } j = 1, \dots, n, \tag{41}$$

$$\sum_{i=1}^{n} a_i x_i = (l_n^{+n}, u_n^n) \sum_{i=1}^{n} b_i y_i. \tag{42}$$

(iii) *$\Gamma_1^{\mathcal{Z}}$ is an inner FODC of $\mathcal{Z}$ with respect to the left-invariant one-form $\theta_n = -\delta\eta_n$, that is, we have*

$$\mathrm{d}z = (\delta-1)^{-1}(\theta_n z - z\theta_n) \quad \textit{for } z \in \mathcal{Z}. \tag{43}$$

Proof. (i) We carry out the proofs of the second and the fourth relations and work with the dual bases $\{\theta_1, \dots, \theta_{n-1}, \eta_1, \dots, \eta_n\}$ and $\{X_1, \dots, X_{n-1}, Y_1, \dots, Y_n\}$. The two other relations follow by a similar slightly simpler reasoning. Using formulas (35), (37), (39) and the above assumptions we compute

$$\begin{aligned}
\mathrm{d}y_i y_j &= \sum_r \gamma_n\gamma_r^{-1} S(u_i^r)\eta_r S(u_j^n) \\
&= \sum_{r=1}^{n-1}\sum_{k,s=1}^{n} \gamma_n\gamma_r^{-1} S(u_i^r)(S(u_j^k)S(l_r^{+s})l_n^{+n}Z_n, S(u_k^n))\eta_s \\
&\quad + \sum_{s=1}^{n-1}\sum_{k=1}^{n} S(u_i^n)S(u_j^k)(\delta^{-1}-1)((X_s, S(u_k^n))\theta_s + (Y_s, S(u_k^n))\eta_s) \\
&\quad + \sum_{k=1}^{n} S(u_i^n)S(u_j^k)(Z_n, S(u_k^n))\eta_n \\
&= \sum_{k,r,s=1}^{n} \gamma_n\gamma_r^{-1} S(u_i^r)S(u_j^k)\delta^{-1}c^{-1}(l_r^{+s}, S^2(u_k^n))\eta_s \\
&\quad + \sum_{s=1}^{n-1}\sum_{k=1}^{n} S(u_i^n S(u_j^k)(\delta^{-1}-1-\zeta\delta^{-1})(Y_s, S(u_k^n))\eta_s.
\end{aligned}$$

The first sum is treated as in the proof of Proposition 3. In this manner it becomes equal to $c^{-1}\delta^{-1}\sum_{k,m}\hat{R}_{ji}^{mk} y_k\,\mathrm{d}y_m$. Put $\tilde{\zeta} := \zeta\delta^{-1} + 1 - \delta^{-1}$. Since $(Y_s, S(u_k^n)) = \gamma_n\gamma_k^{-1}(Y_s, S^{-1}(u_k^n)) = \gamma_n\gamma_k^{-1}\delta_{ks}$ by (34) and $\gamma_n\gamma_k^{-1}\eta_k = \gamma_n\gamma_k^{-1}\omega(S^{-1}(u_k^n)) = \sum_r S^2(u_k^r)\,\mathrm{d}S(u_r^n)$, the second expression yields

$$\begin{aligned}
&\sum_{k=1}^{n-1} -\tilde{\zeta} S(u_i^n)S(u_j^k)\gamma_n\gamma_k^{-1}\eta_k \\
&\quad = \tilde{\zeta} S(u_i^n)S(u_j^n)\eta_n - \sum_{k,r=1}^{n} \tilde{\zeta} S(u_i^n)S(u_j^k)S^2(u_k^r)\,\mathrm{d}S(u_r^n) \\
&\quad = \tilde{\zeta} y_i y_j \eta_n - \tilde{\zeta} y_i\,\mathrm{d}y_i.
\end{aligned}$$

Putting both terms together we obtain the second relation. In order to prove the fourth relation we proceed in a similar manner. Using the facts that $(S(l_n^{-s})l_n^{+n}Z_n, u_n^k) = \zeta\delta^{-1}(Y_s, u_n^k) = 0$ and $(X_s, u_n^k) = \delta_{ks}$ for $s = 1, \dots, n-1$, we obtain

$$\mathrm{d}y_i x_j = \sum_{r=1}^{n-1}\sum_{k,s=1}^{n} \gamma_n\gamma_r^{-1} S(u_i^r)u_k^j(S(l_r^{+s})l_n^{+n}Z_n, u_n^k)\eta_s$$

$$
\begin{aligned}
&+\sum_{s=1}^{n-1}\sum_{k=1}^{n} S(u_i^n)u_k^j(\delta^{-1}-1)((X_s,u_n^k)\theta_s+(Y_s,u_n^k)\eta_s)\\
&+\sum_{k=1}^{n} S(u_i^n)u_k^j(Z_n,u_n^k)\eta_n\\
=&\sum_{k,m,r,s=1}^{n} c\delta\gamma_n\gamma_r^{-1}S(u_i^r)u_k^j u_s^m(\hat{R}^{-1})_{rn}^{ks}\,\mathrm{d}S^{-1}(u_m^n)\\
&+\sum_{k=1}^{n-1}(\delta^{-1}-1)S(u_i^n)u_k^j\theta_k\\
=&\sum_{k,m,r,s=1}^{n} c\delta\gamma_n\gamma_r^{-1}S(u_i^r)u_r^k u_n^s\bar{\mathbf{r}}(u_s^j,u_k^m)\,\mathrm{d}S^{-1}(u_m^n)\\
&-(\delta^{-1}-1)S(u_i^n)u_n^j\theta_n+\sum_{k=1}^{n}(\delta^{-1}-1)S(u_i^n)u_k^j\theta_k\\
=&\sum_{k,m,r,s=1}^{n} c\delta S(u_i^r)S^2(u_r^k)u_n^s\bar{\mathbf{r}}(u_s^j,S^2(u_k^m))\,\mathrm{d}S^{-1}(S^2(u_m^n))\\
&-(\delta^{-1}-1)y_ix_j\theta_n+(\delta^{-1}-1)y_i\,\mathrm{d}x_j\\
=&\sum_{m,s=1}^{n} c\delta(\grave{R}^{-})_{sm}^{ij}x_s\,\mathrm{d}y_m+(\delta^{-1}-1)(y_i\,\mathrm{d}x_j-y_ix_j\theta_n).
\end{aligned}
$$

(ii) Since $\theta_n=\sum_i y_i\,\mathrm{d}x_i$ and $\eta_n=\sum_i\gamma_n\gamma_i^{-1}x_i\,\mathrm{d}y_i$, the four relations in (i) imply that the set $\{\mathrm{d}x_1,\ldots,\mathrm{d}x_n,\mathrm{d}y_1,\ldots,\mathrm{d}y_n\}$ generates $\Gamma_1^{\mathcal{Z}}$ as a left $\mathcal{Z}$-module. It remains to verify that (40) is equivalent to (41) and (42). Since $X_n=-\delta Y_n$, the element $S^{-1}(u_n^n)+\delta u_n^n$ is annihilated by the whole quantum tangent space $\mathcal{T}_1^{\mathcal{Z}}$ and hence $0=\omega(S^{-1}(u_n^n)+\delta u_n^n)=\eta_n+\delta\theta_n$. Inserting the relations $\eta_n=-\delta\theta_n$, (18) and (28) into (40) we see that (40) reads as

$$\sum_{i=1}^{n}\left(\sum_{r=1}^{n-1}(a_iu_r^i\theta_r+b_i\gamma_n\gamma_r^{-1}S(u_i^r)\eta_r)+(a_iu_n^i-c^2b_iS(u_i^n))\theta_n\right)=0.$$

Since the set $\{\theta_1,\ldots,\theta_n,\eta_1,\ldots,\eta_{n-1}\}$ is a free left $\mathcal{A}$-module basis of $\Gamma_1^{\mathcal{Z}}$ the latter is equivalen to the relations

$$\sum_i a_iu_r^i=\sum_i b_iS(u_i^r)=0\quad\text{for } r=1,\ldots,n-1, \tag{44}$$

$$\sum_i(a_iu_n^i-\delta b_iS(u_i^n))=0. \tag{45}$$

Multiplying $\sum_i a_iu_r^i$ by $S(u_k^r)$ and $\sum_i b_iS(u_i^r)$ by $S^2(u_r^k)=\gamma_k\gamma_r^{-1}u_r^k$ and summing over r, (44) implies (41). Formula (45) is nothing but (42). Using the relations $\sum_i y_ix_i=\sum_i x_iy_i\gamma_i\gamma_n^{-1}=1$, (41) in turn implies (44).

(iii) It suffices to prove (43) for the generators $z = x_i, y_i$. Because $(Y_s, u^j_n) = 0$ and $(X_s, u^j_n) = \delta_{js}$ for $s = 1, \ldots, n-1$, it follows from (38) and (18) that

$$\begin{aligned}
\theta_n x_i &= u^i_n (Z_n, u^n_n)\theta_n + (\delta - 1)\sum_{s=1}^{n-1} u^i_s \theta_s \\
&= u^i_n(\delta - (\delta - 1))\theta_n + (\delta - 1)\sum_{s=1}^{n} u^i_s\theta_s \\
&= x_i\theta_n + (\delta - 1)\,\mathrm{d}x_i,
\end{aligned}$$

which gives (43) in the case $z = x_i$. Similarly, using formulas (39) and (28) we get $\mathrm{d}y_i = (\delta^{-1} - 1)^{-1}(\eta_n y_i - y_i\eta_n)$, so that $\mathrm{d}y_i = (\delta - 1)^{-1}(\theta_n y_i - y_i\theta_n)$. □

Next we turn to the FODC $\Gamma_4^{\mathcal{Z}}$ on $\mathcal{Z}$. We take the linear span $\mathcal{T}_4^{\mathcal{Z}}$ of functionals

$$\begin{aligned}
&X_i := \gamma^{-1} l_i^{-n} l_n^{+n} \quad \text{and} \quad Y_i := \zeta^{-1} S(l_n^{+i}) l_n^{+n}, \qquad i = 1, \ldots, n-1, \\
&X_n = (\delta - 1)^{-1}(Z_n - \varepsilon) \quad \text{and} \quad Y_n = (\delta^{-1} - 1)^{-1}(Z_n - \varepsilon).
\end{aligned}$$

For $i = 1, \ldots, n-1$, we then have

$$\Delta(X_i) - \varepsilon \otimes X_i = \sum_{j=1}^{n-1} X_j \otimes l_i^{-j} l_n^{+n}, \qquad \Delta(X_n) - \varepsilon \otimes X_n = X_n \otimes Z_n, \tag{46}$$

$$\Delta(Y_i) - \varepsilon \otimes Y_i = \sum_{j=1}^{n-1} Y_j \otimes S(l_j^{-i}) l_n^{+n}, \qquad \Delta(Y_n) - \varepsilon \otimes Y_n = Y_n \otimes Z_n, \tag{47}$$

These formulas and the relations $(X_i, u^n_j) = (Y_i, S^{-1}(u^n_j)) = \delta_{ij}$ and $(X_i, S(u^n_k)) = (Y_i, u^k_n) = 0$ for $i, j = 1, \ldots, n$ and $k = 1, \ldots, n-1$ imply that $\mathcal{T}_4^{\mathcal{Z}}$ is the quantum tangent space of a $(2n-1)$-dimensional left-covariant FODC Γ_4 of $\mathcal{A}$. The commutation relations of this FODC between the one-forms θ_r, η_s and elements of $\mathcal{A}$ are

$$\begin{aligned}
\theta_r a &= \sum_{s=1}^{n-1} a_{(1)}(l_s^{-r} l_n^{+n}, a_{(2)})\theta_s, \\
\eta_r a &= \sum_{s=1}^{n-1} a_{(1)}((S(l_r^{+s}) l_n^{+n}, a_{(2)})\eta_s, \\
\theta_n a &= a_{(1)}(Z_n, a_{(2)})\theta_n, \qquad \eta_n a = a_{(1)}(Z_n, a_{(2)})\eta_n
\end{aligned}$$

for $r, s = 1, \ldots, n-1$. Let $\Gamma_4^{\mathcal{Z}}$ denote the FODC of $\mathcal{Z}$ which is induced by the FODC Γ_4 of $\mathcal{A}$. By similar computations as carried out above one proves the following commutation relations of the $\mathcal{Z}$-bimodule $\Gamma_4^{\mathcal{Z}}$:

$$\mathrm{d}x_i x_j = c \sum_{k,m=1}^{n} (\hat{R}^{-1})^{ij})_{km} x_k\,\mathrm{d}x_m - \gamma x_i\,\mathrm{d}x_j + \gamma x_i x_j \theta_n,$$

$$dy_i y_j = c^{-1} \sum_{k,m=1}^{n} \hat{R}^{mk}_{ji} y_k \, dy_m - \zeta y_i \, dy_j + \zeta y_i y_j \eta_n,$$

$$dx_i y_j = c^{-1} \sum_{k,m=1}^{n} \hat{R}^{ki}_{mj} y_k \, dx_m,$$

$$dy_i x_j = c \sum_{k,m=1}^{n} (\hat{R}^{-})^{ij}_{km} x_k \, dy_m.$$

Thes are precisely the relations which are obtained by setting formally $\delta = 1$ in the commutation relations for the FODC $\Gamma_1^{\mathcal{Z}}$ (see Proposition 4(i)). That is, the FODC $\Gamma_4^{\mathcal{Z}}$ can be viewed as the limit of the FODC $\Gamma_1^{\mathcal{Z}}$ as $\delta \to 1$. Note that the FODC $\Gamma_1^{\mathcal{Z}}$ has no direct meaning in the case $\delta = 1$.

By "mixing" the elements of the quantum tangent spaces of the FODC Γ_1 and Γ_4 one obtains two other FODC on $\mathcal{Z}$. We briefly describe the quantum Lie algebras of the corresponding FODC Γ_2 and Γ_3 of $\mathcal{A}$ and the commutation rules of these calculi. Let $\mathcal{T}_2^{\mathcal{Z}}$ be the linear span of functionals

$$X_i := \gamma^{-1}\delta^{-1} l_i^{-n} l_n^{+n} Z_n \quad \text{and} \quad Y_i := \zeta^{-1} S(l_n^{+i}) l_n^{+n}, \qquad i = 1, \ldots, n-1,$$
$$X_n := (\delta - 1)^{-1}(Z_n - \varepsilon) \quad \text{and} \quad Y_n := -\delta X_n = (\delta^{-1} - 1)^{-1}(Z_n - \varepsilon),$$

and $\mathcal{T}_3^{\mathcal{Z}}$ the span of functionals

$$X_i := \gamma^{-1} l_i^{-n} l_n^{+n} \quad \text{and} \quad Y_i := \zeta^{-1} \delta S(l_n^{+i}) l_n^{+n} Z_n, \qquad i = 1, \ldots, n-1,$$
$$X_n := (\delta - 1)^{-1}(Z_n - \varepsilon) \quad \text{and} \quad Y_n := -\delta X_n = (\delta^{-1} - 1)^{-1}(Z_n - \varepsilon).$$

From the formulas (33), (34), (46) and (47) we see that $\mathcal{T}_2^{\mathcal{Z}}$ and $\mathcal{T}_3^{\mathcal{Z}}$ are quantum tangent spaces of $(2n-1)$-dimensional left-covariant FODC Γ_2 and Γ_3 of $\mathcal{A}$, respectively. From these formulas we also read off the following commutation relations between the left-invariant one-forms θ_i, η_k and elements $a \in \mathcal{A}$:

$$\Gamma_2: \quad \theta_r a = \sum_{s=1}^{n-1} a_{(1)}(l_s^{-r} l_n^{+n} Z_n, a_{(2)})\theta_s, \qquad \eta_s a = \sum_{s=1}^{n-1} a_{(1)}(S(l_r^{+s}) l_n^{+n}, a_{(2)})\eta_s,$$

$$\theta_n a = a_{(1)}(Z_n, a_{(2)})\theta_n + (\delta - 1)\sum_{s=1}^{n-1} a_{(1)}(X_s, a_{(2)})\theta_s,$$

$$\eta_n a = a_{(1)}(Z_n, a_{(2)})\eta_n + (\delta^{-1} - 1)\sum_{s=1}^{n-1} a_{(1)}(X_s, a_{(2)})\theta_s,$$

$$\Gamma_3: \quad \theta_r a = \sum_{s=1}^{n-1} a_{(1)}(l_s^{-r} l_n^{+n}, a_{(2)})\theta_s, \qquad \eta_s a = \sum_{s=1}^{n-1} a_{(1)}(S(l_r^{+s}) l_n^{+n} Z_n, a_{(2)})\eta_s,$$

$$\theta_n a = a_{(1)}(Z_n, a_{(2)})\theta_n + (\delta - 1)\sum_{s=1}^{n-1} a_{(1)}(Y_s, a_{(2)})\eta_s,$$

$$\eta_n a = a_{(1)}(Z_n, a_{(2)})\eta_n + (\delta^{-1} - 1)\sum_{s=1}^{n-1} a_{(1)}(Y_s, a_{(2)})\eta_s,$$

where $r = 1, \ldots, n-1$. As earlier, the FODC on $\mathcal{Z}$ induced by the FODC Γ_j on $\mathcal{A}$ is denoted by $\Gamma_j^{\mathcal{Z}}$, $j = 2, 3$. From the preceding set of formulas one gets the following commutation rules for the $\mathcal{Z}$-bimodule $\Gamma_j^{\mathcal{Z}}$:

$$\Gamma_2^{\mathcal{Z}}: \quad \mathrm{d}x_i x_j = c\delta \sum_{k,m=1}^{n} (\hat{R}^{-1})^{ij}_{km} x_k \,\mathrm{d}x_m + (\delta - \gamma\delta - 1)(x_i \,\mathrm{d}x_j - x_i x_j \theta_n),$$

$$\mathrm{d}y_i y_j = c^{-1} \sum_{k,m=1}^{n} \hat{R}^{mk}_{ji} y_k \,\mathrm{d}y_m - \zeta y_i \,\mathrm{d}y_j + \zeta y_i y_j \eta_n,$$

$$\mathrm{d}x_i y_j = c^{-1}\delta^{-1} \sum_{k,m=1}^{n} \hat{R}^{ki}_{mj} y_k \,\mathrm{d}x_m,$$

$$\mathrm{d}y_i x_j = c \sum_{k,m=1}^{n} (\grave{R}^{-})^{ij}_{km} x_k \,\mathrm{d}y_m + (\delta^{-1} - 1) y_i \,\mathrm{d}x_j$$

$$\Gamma_3^{\mathcal{Z}}: \quad \mathrm{d}x_i x_j = c \sum_{k,m=1}^{n} (\hat{R}^{-1})^{ij}_{km} x_k \,\mathrm{d}x_m - \gamma x_i \,\mathrm{d}x_j + \gamma x_i x_j \theta_n,$$

$$\mathrm{d}y_i y_j = c^{-1}\delta^{-1} \sum_{k,m=1}^{n} \hat{R}^{mk}_{ji} y_k \,\mathrm{d}y_m + (\delta^{-1} - \zeta\delta^{-1} - 1)(y_i \,\mathrm{d}y_j - y_i y_j \eta_n),$$

$$\mathrm{d}x_i y_j = c^{-1} \sum_{k,m=1}^{n} \hat{R}^{ki}_{mj} y_k \,\mathrm{d}x_m + (\delta - 1) x_i \,\mathrm{d}y_j,$$

$$\mathrm{d}y_i x_j = c\delta \sum_{k,m=1}^{n} (\grave{R}^{-})^{ij}_{km} x_k \,\mathrm{d}y_m.$$

Recall that by Proposition 4(iii) the FODC $\Gamma_1^{\mathcal{Z}}$ of $\mathcal{Z}$ is inner. It turns out that none of the three other FODC $\Gamma_2^{\mathcal{Z}}, \Gamma_3^{\mathcal{Z}}, \Gamma_4^{\mathcal{Z}}$ is inner. Indeed, from the above commutation rules one easily derives that

$$\Gamma_2^{\mathcal{Z}}: \theta_n y_i = \delta^{-1} y_i \theta_n, \qquad \Gamma_3^{\mathcal{Z}}: \quad \theta_n x_i = x_i \theta_n, \qquad \Gamma_4^{\mathcal{Z}}: \theta_n y_i = y_i \theta_n \tag{48}$$

for all $i = 1, \ldots, n$. Further, for all four FODC $\Gamma_j^{\mathcal{Z}}$ we have $\theta_n = -\delta\eta_n$ and this is up to complex multiples the only left-invariant one-form of $\Gamma_j^{\mathcal{Z}}$. Therefore, we conclude at once from (48) that none of the FODC $\Gamma_j^{\mathcal{Z}}$, $j = 2, 3, 4$, of $\mathcal{Z}$ is inner.

All four left-covariant FODC $\Gamma_j^{\mathcal{Z}}$ of $\mathcal{Z}$ depend on the group-like element $Z_n \in \mathcal{A}^{\circ}$. It can be freely choosen such that it satisfies the conditions (31) and (32). This dependence is reflected by the appearance of the parameter $\delta = (Z_n, u^n_n)$ in the above formulas. For the FODC $\Gamma_1^{\mathcal{Z}}$ a distinguished choice of Z_n is $Z_n = (l_n^{-n})^2$. In this case $\mathcal{T}_1^{\mathcal{Z}}$ is just the sum of the quantum tangent spaces $\mathcal{T}^{\mathcal{X}}$ and $\mathcal{T}^{\mathcal{Y}}$ considered in Sections 2 and 3 and the FODC

$\Gamma_1^{\mathcal{Z}}$ might be thought as gluing together the FODC $\Gamma^{\mathcal{X}}$ and $\Gamma^{\mathcal{Y}}$. Further, if we assume in addition the conditions (11) and (23), then we have $\alpha = \gamma c^{-2} = \delta - 1, \beta = \gamma c^2 = \delta^{-1} - 1$, and $\delta = c^{-2}$, so that $\gamma\delta + 1 - \delta = \zeta\delta^{-1} + 1 - \delta^{-1} = 0$. Thus, in this case the first two relations for the FODC $\Gamma_1^{\mathcal{Z}}$ in Proposition 4(i) become even linear.

5. Application to the quantum homogeneous space $GL_q(N)/GL_q(N-1)$

In this section let $\mathcal{A}$ denote the Hopf algebra $\mathcal{O}(GL_q(N))$, $\mathbf{u} = (u^i_j)_{i,j=1,\dots,N}$ the fundamental corepresentation of $\mathcal{A}$ and $\hat{R}$ the corresponding R-matrix given by (see [5])

$$R^{ji}_{kl} \equiv \hat{R}^{ij}_{kl} := q^{\delta_{ij}}\delta_{il}\delta_{jk} + (q - q^{-1})\theta(j-i)\delta_{ik}\delta_{jl}, \qquad i,j,k,l = 1,\dots,N. \quad (49)$$

The Hopf algebra $\mathcal{A}$ is coquasitriangular with universal r-form $\mathbf{r}$ determined by

$$\mathbf{r}(u^i_j, u^k_l) = \hat{R}^{ki}_{jl}, \quad i,j,k,l = 1,\dots,N. \quad (50)$$

Further, we suppose that Z_n is a monomial in the main diagonal L-functionals $l_i^{\pm i}$.

Using (49) and (50) one easily verifies that the above assumptions (8), (10), (11), (19), (21)–(23) and (29)–(32) are then fulfilled with $n = N, \alpha = -\zeta = q^{-2}-1, \beta = -\gamma = q^2 - 1$, $c = q$ and $\gamma_i = q^{2i}$. Therefore, *all results obtained in Sections* 3–5 *are valid in this case.* Here we shall add only a few remarks concerning these results rather than restating them in the present situation. The quantum homogeneous space $\mathcal{X}$ is then, of course, isomorphic to the quantum vector space $\mathcal{O}(\mathbb{C}^N_q)$ [6, Proposition 9.11]) and the FODC $\Gamma^{\mathcal{X}}$ is one of the two well-known covariant calculi on $\mathcal{O}(\mathbb{C}^N_q)$ discovered in [11,17]. However, the approach given in Section 2 might still be of interest. The FODC $\Gamma_j^{\mathcal{Z}}$, $j = 1,2,3,4$, developed in Section 4 are left-covariant FODC on the subalgebra $\mathcal{Z}$ of $\mathcal{A}$ generated by the element $x_i \equiv u^i_N$ and $y_i \equiv S(u^N_i)$, $i = 1,\dots,N$. All four FODC have the property that $\Gamma_j^{\mathcal{Z}}$ as a left $\mathcal{Z}$-module is generated by the differentials $\mathrm{d}x_1,\dots,\mathrm{d}x_N, \mathrm{d}y_1,\dots,\mathrm{d}y_N$. The FODC $\Gamma_1^{\mathcal{Z}}$ described by Proposition 4 is inner. In the special case $Z_n = (l_n^{-n})^2$ it coincides with the distinguished calculus considered in [16] (more precisely with its left-covariant counter-part).

The importance of the left quantum space $\mathcal{Z}$ stems from the fact that it is (isomorphic to) the quantum homogeneous space $GL_q(N)/GL_q(N-1)$. Indeed, there is a unique surjective Hopf algebra homomorphism $\pi : GL_q(N) \to GL_q(N-1)$ such that

$$\pi(u^i_j) = w^i_j, \quad i,j = 1,\dots,N-1,$$
$$\pi(u^i_N) = \pi(u^N_i) = 0, \quad i = 1,\dots,N-1, \qquad \pi(u^N_N) = 1,$$

where w^i_j, $i,j = 1,\dots,N-1$, denote the matrix entries of the fundamental matrix for the quantum group $GL_q(N-1)$. Then the set

$$\mathcal{O}(GL_q(N)/GL_q(N-1)) := \{a \in \mathcal{O}(GL_q(N)) : (\mathrm{id} \otimes \pi) \circ \Delta(a) = a \otimes 1\}$$

of all right $GL_q(N-1)$-invariant elements of $\mathcal{O}(GL_q(N))$ is a subalgebra and a left quantum space for $\mathcal{O}(GL_q(N))$ with respect to the coaction $\Delta\lceil\mathcal{O}(GL_q(N)/GL_q(N-1))$. The elements x_i and y_i are in $\mathcal{O}(GL_q(N)/GL_q(N-1))$, so that $\mathcal{Z} \subseteq \mathcal{O}(GL_q(N)/GL_q(N-1))$.

If q is not a root of unity, then we have the equality $\mathcal{Z} = \mathcal{O}(GL_q(N)/GL_q(N-1))$. (For the corresponding right quantum space $GL_q(N-1)\backslash GL_q(N)$ this is proved in [8, Proposition 4.4], or [6, Section 14.6]. The proof for the left quantum space $GL_q(N)/GL_q(N-1)$ is completely analogous.)

Suppose now that q is a real number and $q \neq 0, \pm 1$. Then it is well known that the Hopf algebra $\mathcal{O}(GL_q(N))$ is a Hopf $*$-algebra, denoted by $\mathcal{O}(U_q(N))$, with involution determined by $(u^i_j)^* = S(u^j_i)$, $i, j = 1, \dots, N$. Further, the algebra $\mathcal{O}(GL_q(N)/GL_q(N-1))$ is a $*$-subalgebra such that $x_i^* \equiv (u^i_N)^* = y_i \equiv S(u^N_i)$ and a left $*$-quantum space for $\mathcal{O}(U_q(N))$. It is denoted by $\mathcal{O}(U_q(N)/U_q(N-1))$ and called the coordinate $*$-algebra of the *quantum sphere* associated with the quantum group $U_q(N)$. In this case the two left-covariant FODC Γ_1 and Γ_4 of $\mathcal{O}(U_q(N))$ and hence their induced FODC $\Gamma_1^{\mathcal{Z}}$ and $\Gamma_4^{\mathcal{Z}}$ on $\mathcal{Z} = \mathcal{O}(U_q(N)/U_q(N-1))$ are $*$-calculi. We prove these assertions for Γ_1 and $\Gamma_1^{\mathcal{Z}}$. First note that $(l^{\pm i}_j)^* = S(l^{\mp j}_i)$ (see [6, formula (10.47)]) for the corresponding involution of the Hopf dual $\mathcal{O}(GL_q(N))^\circ$. Hence we obtain $X_N^* = X_N$ and $X_i^* = (l_i^{-N} l_N^{-N} Z_N)^* = Z_N l_N^{-N} S(l_N^{+i})$ for $i = 1, \dots, N-1$. Since Z_N is a monomial in the L-functionals $l_i^{\pm i}$, $Z_N l_N^{-N} S(l_N^{+i})$ is a complex multiple of $S(l_N^{+i}) l_N^{-N} Z_N = Y_i$. Therefore, we have $X^* \in \mathcal{T}_1^{\mathcal{Z}}$ for all $X \in \mathcal{T}_1^{\mathcal{Z}}$, so that $\Gamma_1^{\mathcal{Z}}$ is a $*$-calculus of $\mathcal{O}(U_q(N))$ by Proposition 14.6 in [6]. Since $\mathcal{Z}$ is a $*$-subalgebra of $\mathcal{O}(U_q(N))$, the induced FODC $\Gamma_1^{\mathcal{Z}}$ and $\Gamma_1^{\mathcal{Z}}$ is also a $*$-calculus. Thus, *the FODC $\Gamma_1^{\mathcal{Z}}$ and $\Gamma_4^{\mathcal{Z}}$ are $*$-calculi on the coordinate $*$-algebra $\mathcal{Z} = \mathcal{O}(U_q(N)/U_q(N-1))$ of the quantum sphere*. Note that because these FODC are $*$-calculi it suffices to prove only one of the commutation relations for $\mathrm{d}x_i x_j$ and $\mathrm{d}y_i y_j$ and one of the relations for $\mathrm{d}x_i y_j$ and $\mathrm{d}y_i x_j$. The two others follow then by applying the involution and inverting the corresponding R-matrix. The FODC $\Gamma_2^{\mathcal{Z}}$ and $\Gamma_3^{\mathcal{Z}}$ are not $*$-calculi on $\mathcal{Z}$, but one has $(\mathcal{T}_2^{\mathcal{Z}})^* = \mathcal{T}_3^{\mathcal{Z}}$.

Let us return to the general case where q is a complex number such that $q \neq 0, \pm 1$. From its very construction it is clear that the left-covariant $(2n-1)$-dimensional FODC Γ_1 of the Hopf algebra $\mathcal{O}(GL_q(N))$ is a useful tools for the study of the induced FODC $\Gamma_1^{\mathcal{Z}}$ on the subalgebra $\mathcal{Z}$. However, Γ_1 is not suitable as a FODC of the Hopf algebra $\mathcal{O}(GL_q(N))$ itself, because the generators X_i, Y_i of the quantum tangent space $\mathcal{T}^{\mathcal{Z}}$ are only supported on the last row and column of the fundamental matrix $\mathbf{u} = (u^i_j)$. To remedy this defect, one can construct an N^2-dimensional left-covariant FODC Γ on $\mathcal{A} = \mathcal{O}(GL_q(N))$ that induces the FODC $\Gamma_1^{\mathcal{Z}}$ on $\mathcal{Z}$ as well. We restrict ourselves to the distinguished calculus $\Gamma_1^{\mathcal{Z}}$ with $Z_n = (l_n^{-n})^2$. Let $\mathcal{T}$ be the linear span of linear functionals

$$X_{ij} = (q^{-2} - 1)^{-1} l_i^{-j} l_j^{-j}, \quad i < j, \tag{51}$$

$$Y_{ji} = (q^2 - 1)^{-1} S(l_j^{+i}) l_j^{-j}, \quad i < j, \tag{52}$$

$$X_{ii} = (q^{-2} - 1)^{-1} ((l_i^{-i})^2 - \varepsilon), \qquad Y_{ii} = -q^{-2} X_{ii} \tag{53}$$

on $\mathcal{A}$. For $i \leq j$, $i, j = 1, \dots, N$, we then have

$$\Delta(X_{ij}) - \varepsilon \otimes X_{ij} = \sum_{k \leq j} X_{kj} \otimes l_i^{-k} l_j^{-j}, \tag{54}$$

$$\Delta(Y_{ji}) - \varepsilon \otimes Y_{ji} = \sum_{k \le j} Y_{jk} \otimes S(l_k^{+i}) l_j^{-j}, \tag{55}$$

Thus, by Lemma 1, there is a left-covariant FODC Γ of $\mathcal{A}$ which has the quantum tangent space $\mathcal{T}$. From the explicit form (49) of the matrix $\hat{R}$ and its inverse $\hat{R}^{-1} = (q - q^{-1})\hat{R} + I$ we compute

$$(X_{ij}, u_s^r) = \delta_{ir}\delta_{js} \quad \text{and} \quad (Y_{ji}, S^{-1}(u_r^s)) = \delta_{ir}\delta_{js} \ \text{ for } i \le j. \tag{56}$$

Setting $\theta_{ij} := \omega(u_j^i) = \sum_k S(u_k^i)\,\mathrm{d}u_j^k$ and $\eta_{ji} := \omega(S^{-1}(u_i^j)) = \sum_k u_i^k\,\mathrm{d}S^{-1}(u_k^j)$ for $i \le j$, the formulas (6) and (56) imply that

$$(X_{ij}, \theta_{rs}) = (Y_{ji}, \eta_{sr}) = \delta_{ir}\delta_{js} \quad \text{for } i \le j, i, j, r, s = 1, \ldots, N. \tag{57}$$

In particular, the functionals $X_{ij}, Y_{rs}, i \le j, s < r$, are linearly independent, so that the FODC Γ has dimension N^2. Further, it follows from (6) and (57) that the sets $\{\theta_{ij}, \eta_{rs}, i \le j, s < r\}$ and $\{X_{ij}, Y_{rs}; i \le j, s < r\}$ and also the sets $\{\theta_{ij}, \eta_{rs}; i < j, s \le r\}$ and $\{X_{ij}, Y_{rs}; i < j, s \le r\}$ are dual bases of ${}_{\mathrm{inv}}\Gamma$ and $\mathcal{T}$, respectively. It is not difficult to verify that the two calculi Γ and Γ_1 with $Z_n = (l_n^{-n})^2$ of $\mathcal{A}$ induce the same FODC $\Gamma_1^{\mathcal{Z}}$ on the quantum space $\mathcal{Z}$.

For $j = 1, \ldots, N$, let $\mathcal{T}_j$ denote the linear span of functionals X_{ij} and $Y_{ji}, i \le j$. From (54) and (55) we conclude that there is a $(2j-1)$-dimensional left-covariant FODC Γ^j on $\mathcal{O}(GL_q(N))$ which has the quantum tangent space $\mathcal{T}_j$. The FODC Γ^N is nothing but the FODC Γ_1 developed in Section 4 (as always throughout this discussion, with $Z_n = (l_n^{-n})^2$). Since the linear quantum tangent space $\mathcal{T}$ is the direct sum of vector spaces $\mathcal{T}_1, \ldots, \mathcal{T}_N$, the FODC is the direct sum of FODC $\Gamma^1, \ldots, \Gamma^N$. These and other properties indicate that the FODC Γ^i is a promising tool for the study of the interplay between the quantum group $GL_q(N)$ and the quantum homogeneous spaces $GL_q(j)/GL_q(j-1)$, $j = 2, \ldots, N$. The FODC Γ is only left-covariant, but not bicovariant. However, because of its particular and simple structure the FODC Γ might be even more important and useful for appliations and computations than the bicovariant calculi of the Hopf algebra $\mathcal{O}(GL_q(N))$. We shall return to this matter in Section 7.

At the end of this section, let us briefly turn to the quantum group $SL_q(N)$. The Hopf algebra $\mathcal{O}(SL_q(N))$ is also quasitriangular with universal r-form $\mathbf{r}$ such that

$$\mathbf{r}(u_j^i, u_l^k) = z\hat{R}_{jl}^{ki}, \quad i, j, k, l = 1, \ldots, N, \tag{58}$$

where $\hat{R}$ is given by (49) and z is a complex Nth root of q^{-1}. Then the linear span of functionals $X_{ij}, Y_{ji}, i < j$, and $X_{rr}, r = 2, \ldots, N$, defined by (51)–(53) is also the quantum tangent space of a $(N^2 - 1)$-dimensional FODC on $\mathcal{O}(SL_q(N))$. It should be emphasized that because of the appearance of the number z in (58) the equalities (56) are no longer valid for $\mathcal{O}(SL_q(N))$. Some $(N^2 - 1)$-dimensional left-covariant FODC on $\mathcal{O}(SL_q(N))$ with reasonable properties have been constructed in [14]. This FODC is different from those in [14], but is based on a similar idea.

6. A left-covariant FODC on $GL_q(N)/GL_q(N-1)$ induced from a bicovariant FODC on $GL_q(N)$

In this section we retain the notation of Section 5. Let Γ_{bi} be the bicovariant FODC on $\mathcal{A} = \mathcal{O}(GL_q(N))$ constructed by the bicovariant bimodule $(u^c \otimes u, L^+ \otimes L^{-,c})$. (Details can be found, for instance, in [6, Sections 14.5 and 14.6]). Here we only need the two facts (see [6, 14.6.3 and Example 14.8]) that the set $\{\omega_{ij} := \omega(u^i_j) = \sum_k S(u^i_k)\,\mathrm{d}u^k_j, i, j = 1, \ldots, N\}$ is a basis of the vector space $_{\mathrm{inv}}(\Gamma_{\mathrm{bi}})$ of left-invariant one-forms of Γ_{bi} and that the commutation rules between the forms ω_{ij} and an element $a \in \mathcal{A}$ are given by

$$\omega_{ij} a = \sum_{r,s} a_{(1)} l^{+i}_r (a_{(2)}) S(l^{-s}_j)(a_{(3)}) \omega_{rs}. \tag{59}$$

Proposition 5. *The FODC Γ_{bi} induces a left-covariant FODC $\Gamma^{\mathcal{Z}}$ on the quantum space $\mathcal{Z}$ such that*

$$\mathrm{d}x_i x_j = q \sum_{k,m} \hat{R}^{ij}_{km} x_k\,\mathrm{d}x_m, \qquad \mathrm{d}y_i y_j = q^{-1} \sum_{k,m} (\hat{R}^{-1})^{ji}_{mk} y_k\,\mathrm{d}y_m,$$

$$\mathrm{d}x_i y_j = q^{-1} \sum_{k,m} (\hat{R}^{-1})^{ki}_{mj} y_k\,\mathrm{d}x_m, \qquad \mathrm{d}y_i x_j = q \sum_{k,m} \grave{R}^{ij}_{km} x_k\,\mathrm{d}y_m,$$

where $\grave{R}^{ij}_{km} := \mathbf{r}(S^2(u^k_j), u^i_m)$, $i, j, k, m = 1, \ldots, N$. Further, we have $\omega_{NN} x_i = q^2 x_i \omega_{NN}$ and $\omega_{NN} y_i = q^{-2} y_i \omega_{NN}$ for $i = 1, \ldots, N$.

Proof. We verify, for instance, the third commutation relation. From the explicit form (49) of the matrix $\hat{R}$ it follows that $(\hat{R}^{-1})^{sN}_{lN} = q^{-1}\delta_{sN}\delta_{lN}$ for $s, l = 1, \ldots, N$. Using essentially this fact and formula (59) we compute

$$\begin{aligned}
\mathrm{d}x_i y_j &= \sum_k u^i_k \omega_{kN} S(u^N_j) \\
&= \sum_{k,m,l,r,s} u^i_k S(u^m_j)(l^{+k}_r, S(u^l_m))(S(l^{-s}_N), S(u^N_l))\omega_{rs} \\
&= \sum_{r,s,l} \left(\sum_{k,m} u^i_k S(u^m_j)(\hat{R}^{-1})^{lk}_{rm} \right) q^{2N-2l} (\hat{R}^{-1})^{sN}_{lN}\, \omega_{rs} \\
&= \sum_{r,s,l} \left(\sum_{k,m} S(u^l_k) u^m_r (\hat{R}^{-1})^{ki}_{mj} \right) q^{2N-2l} q^{-1} \delta_{sN}\delta_{lN}\omega_{rs} \\
&= \sum_{k,m,r} q^{-1} (\hat{R}^{-1})^{ki}_{mj} S(u^N_k) u^m_r \omega_{rN} \\
&= \sum_{k,m} q^{-1} (\hat{R}^{-1})^{ki}_{mj} y_k\,\mathrm{d}x_m.
\end{aligned}$$

The other relations follow by similar reasonings as above or as used earlier. We shall not carry out the details. □

The FODC $\Gamma^{\mathcal{Z}}$ is another good candidate of a reasonable differential calculus on the quantum homogeneous space $\mathcal{Z}$. It is a $*$-calculus if q is real and the involution of $\mathcal{Z}$ is given by $x_i^* = y_i$, $i = 1, \ldots, N$, because the FODC Γ_{bi} on $\mathcal{O}(GL_q(N))$ is known to be a $*$-calculus with respect to the involution $(u_j^i)^* = S(u_i^j)$, $i, j = 1, \ldots, N$. But there is a striking difference between the two distinguished calculi $\Gamma^{\mathcal{Z}}$ and $\Gamma_1^{\mathcal{Z}}$: $\Gamma_1^{\mathcal{Z}}$ is inner, while $\Gamma^{\mathcal{Z}}$ is not. In order to verify the latter, it suffices to note that $\omega_{NN} x_i - x_i \omega_{NN} = (q^2 - 1) x_i \omega_{NN}$ is obviously not a multiple of $\mathrm{d}x_i$.

7. A recipe for the construction of left-covariant FODC

The first order differential calculi on quantum homogeneous spaces developed above are induced from left-covariant calculi on the quantum group. All these left-covariant calculi on the corresponding Hopf algebra are built by the same simple recipe that will be elaborated more explicitly in this section. As always, $\mathcal{A}$ is a coquasitriangular Hopf algebra and $l_j^{\pm i}$ are the L-functionals on $\mathcal{A}$ with respect to a fixed corepresentation $\mathbf{u} = (u_j^i)_{i,j=1,\ldots,n}$ of $\mathcal{A}$. Throughout this section we retain assumption (19).

Let $i, j \in \{1, \ldots, n\}$ be two indices such that $i \leq j$ and let Z be a group-like element of $\mathcal{A}^\circ$. Define

$$\begin{aligned}
X_r^+ &= l_r^{+i} l_i^{-i} Z, \quad r = i+1, \ldots, j, \quad \text{and} \quad X_i^+ = Z - \varepsilon,\\
X_r^- &= l_r^{-j} l_j^{+j} Z, \quad r = i, \ldots, j-1, \quad \text{and} \quad X_j^- = Z - \varepsilon,\\
Y_r^+ &= S(l_j^{+r}) l_j^{+j} Z, \quad r = i, \ldots, j-1, \quad \text{and} \quad Y_j^+ = Z - \varepsilon,\\
Y_r^- &= S(l_i^{-r}) l_i^{-i} Z, \quad r = i+1, \ldots, j, \quad \text{and} \quad Y_i^- = Z - \varepsilon,\\
\mathcal{T}_{ij}^{\pm}(Z) &= \mathrm{Lin}\{X_r^{\pm}; i \leq r \leq j\},\\
\mathcal{T}_{ji}^{\pm}(Z) &= \mathrm{Lin}\{Y_r^{\pm}; i \leq r \leq j\}.
\end{aligned}$$

Using (19) one easily verifies that each vector space $\mathcal{T} = \mathcal{T}_{ij}^{\pm}(Z)$ has the properties that $X(1) = 0$ and $\Delta(X) - \varepsilon \otimes X \in \mathcal{T} \otimes \mathcal{A}^\circ$ for all $X \in \mathcal{T}$. Hence, by Lemma 1 each space $\mathcal{T}_{ij}^{\mp}(Z), \mathcal{T}_{ji}^{\pm}(Z)$ is the quantum tangent space of a left-covariant FODC $\Gamma_{ij}^{\pm}$, $\Gamma_{ji}^{\pm}$ on $\mathcal{A}$. Let us call the first order calculi of the form $\Gamma_{ij}^{\pm}$, $\Gamma_{ji}^{\pm}$ *elementary* FODC. All left-covariant FODC on $\mathcal{A}$ occurring in this paper are diret sums of elementary FODC (with possible different group-like elements Z!). By forming sums of elementary FODC one gets a large supply of left-covariant FODC which have a very simple structure and are easy to handle. FODC of this form have been introduced in [13]. Note that the commutation rules of the elements of the quantum tangent spaces obtained in this manner are not necessarily quadratically closed and that the dimensions of the spaces of higher forms may be different from the corresponding classical dimensions (see [13] for such examples).

For the group-like elements Z one may take, for instance, a monomial in the main diagonal L-functionals $l_i^{\pm i}$, $i = 1, \ldots, n$. Interesting choices of Z are, of course, $Z = l_i^{\pm i}$ for $\mathcal{T}_{ij}^{\pm}(Z)$

and $Z = l_i^{\mp i}$ for $\mathcal{T}_{ji}^{\pm}(Z)$ or $Z = -\delta_{ij}\varepsilon$; for all four FODC. Let us illustrate this by simple examples and set

$$\mathcal{T}^+ = \sum_i \mathcal{T}_{in}^+(l_i^{+i}) = \mathrm{Lin}\{l_j^{+i} - \delta_{ij}\varepsilon;\ i \le j, i, j = 1, \ldots, n\},$$

$$\mathcal{T}^- = \sum_j \mathcal{T}_{1j}^+(l_j^{-j}) = \mathrm{Lin}\{l_i^{-j} - \delta_{ij}\varepsilon;\ i \le j, i, j = 1, \ldots, n\},$$

$$\mathcal{T}_+ = \sum_j \mathcal{T}_{j1}^+(l_j^{-j}) = \mathrm{Lin}\{S(l_j^{+i}) - \delta_{ij}\varepsilon;\ i \le j, i, j = 1, \ldots, n\},$$

$$\mathcal{T}_- = \sum_i \mathcal{T}_{ni}^-(l_i^{+i}) = \mathrm{Lin}\{S(l_i^{-j}) - \delta_{ij}\varepsilon;\ i \le j, i, j = 1, \ldots, n\}.$$

Then, $\mathcal{T}^+, \mathcal{T}^-, \mathcal{T}_+, \mathcal{T}_-, \mathcal{T}^+ + \mathcal{T}_-$ and $\mathcal{T}^- + \mathcal{T}_+$ are quantum tangent spaces of left-covariant FODC on $\mathcal{A}$.

Now we want to be more specific and suppose that $\mathcal{A}$ is one of the Hopf algebras $\mathcal{O}(G_q)$, $G_q = GL_q(N), SL_q(N), O_q(N), Sp_q(N)$, and $\mathbf{u}$ is the fundamental corepresentation.

Case 1. $\mathcal{A} = \mathcal{O}(GL_q(N))$. Then the vector spaces $\mathcal{T}^+ + \mathcal{T}_-$ and $\mathcal{T}^- + \mathcal{T}_+$ defined above are the quantum tangent spaces of two N^2-dimensional left-covariant FODC on $\mathcal{O}(GL_q(N))$. It is easily seen that the commutation relations of the elements of both quantum tangent spaces are quadratically closed. Further, it can be shown that the dimensions of the spaces of k-forms for the associated universal higher order differential calculi (see [6, 14.3], for this notion) are $\binom{N^2}{k}$ as in the classical case.

Case 2. $\mathcal{A} = \mathcal{O}(SL_q(N))$. In this case, $\mathcal{T}^+ + \mathcal{T}_-$ and $\mathcal{T}^- + \mathcal{T}_+$ are also N^2-dimensional FODC on $\mathcal{O}(SL_q(N))$, but we are interested in FODC that have the classical group dimension $N^2 - 1$. It is rather easy to construct such an FODC: Let $\mathcal{T}_{\mathrm{od}}$ be the sum of $\mathcal{T}_{in}^+(\varepsilon), \mathcal{T}_{1i}^-(\varepsilon), i = 1, \ldots, n$, and let $\mathcal{T}_{\mathrm{md}}$ be the vector space spanned by $N - 1$ of the N functionals $l_i^{+i} - \varepsilon$. Then, $\mathcal{T} = \mathcal{T}_{\mathrm{od}} + \mathcal{T}_{\mathrm{md}}$ is the quantum tangent space of an $(N^2 - 1)$-dimensional FODC on $\mathcal{O}(SL_q(N))$. This first order calculus strongly resembles the ordinary differential calculus on the Lie group $SL(N)$ in many aspects. But it has the disadvantage that the commutation rules between elements of the quantum tangent space (for instance, $l_N^{+i} l_i^{-i}$ and $l_j^{-N} l_N^{+N}$) do not close quadratically. (N^2-1)-dimensional FODC on $\mathcal{O}(SL_q(N))$ that do not have this defect have been constructed in [14]. However, using the same idea as in [14], the quantum tangent space $\mathcal{T}$ can be modified by multiplying the secondary diagonal elements such that commutation relations close quadratically.

In order to be more precise, let f_i and $g_i, i = 1, \ldots, N$, be monomials in the main diagonal L-functionals $l_j^{\pm j}$. Let $\mathcal{T}_{\mathrm{od}}$ be the linear span of $X_{ij} := l_j^{+i} l_i^{-i} f_i$ and $X_{ji} := l_i^{-j} l_j^{+j} g_j, i < j$, and let $\mathcal{T}_{\mathrm{md}}$ be an $(N - 1)$-dimensional vector space generated by functionals of the form $f - \varepsilon$, where f is a monomial in $l_i^{\pm i}, i = 1, \ldots, N$. Suppose that $f_i, g_i \in \mathbb{C}\varepsilon \oplus \mathcal{T}_{\mathrm{md}}$ for $i = 1, \ldots, N$. Then one easily verifies that $\mathcal{T} := \mathcal{T}_{\mathrm{od}} + \mathcal{T}_{\mathrm{md}}$ is the quantum tangent space of an $(N^2 - 1)$-dimensional FODC on $\mathcal{O}(SL_q(N))$. Further, the commutation relations for elements of $\mathcal{T}$ are quadratically closed if and only if $f_i^{-1}g_i(l_i^{+i})^2$ is independent of $i = 1, \dots, N$. (This assertion and the explicit form of commutation rules can be derived from the relations $L_1^{\pm}L_2^{\pm}R = RL_2^{\pm}L_1^{\pm}$ and $L_1^{-}L_2^{+}R = RL_2^{+}L_1^{-}$ using (49). We omit the details.) These conditions can be fulfilled as follows: Fix an index $k \in \{1, \dots, N\}$ and set $g_i = (l_i^{-i})^2(l_k^{+k})^2$ and $f_i = \varepsilon$ for $i = 1, \dots, N$. Another possible choice is $f_i = (l_i^{+i})^2(l_k^{-k})^2$ and $g_i = \varepsilon$ for $i = 1, \dots, N$. These two special cases are in fact the two FODC Γ_1 and Γ_2 constructed in [14].

In order to come into contact with the considerations in Sections 4 and 5, we carry out the same consideration based on the generators X_r^-, Y_r^+ rather than X_r^+, X_r^-. We suppose that the elements f_i, g_i and the vector space $\mathcal{T}_{\mathrm{md}}$ satisfy the assumptions stated in the first half of the preceding paragraph. Now let $\mathcal{T}_{\mathrm{od}}$ be the vector space generated by the functionals $X_{ji} = l_j^{-i}l_i^{+i}f_i$ and $X_{ij} = S(l_j^{+i})l_j^{+j}g_j$, $i < j$. Then $\mathcal{T} := \mathcal{T}_{\mathrm{od}} + \mathcal{T}_{\mathrm{md}}$ is again the quantum tangent space of an (N^2-1)-dimensional FODC on $\mathcal{O}(SL_q(N))$. The commutation relations for $\mathcal{T}$ close quadratically if and only if $f_ig_i(l_i^{+i})^2$ does not depend on $i = 1, \dots, N$.

Case 3 : $\mathcal{A} = \mathcal{O}(O_q(N))$ and $\mathcal{A} = \mathcal{O}(Sp_q(N))$. In this case the fundamental matrix $\mathbf{u}$ fulfills the metric condition

$$\mathbf{u}C\mathbf{u}^tC^{-1} = C\mathbf{u}^tC^{-1}u = I \tag{60}$$

and the R-matrix is given by

$$\hat{R}^{ji}_{mn} = q^{\delta_{ij}-\delta_{ij'}}\delta_{im}\delta_{jn} + (q-q^{-1})\theta(i-m)(\delta_{jm}\delta_{in} - \epsilon C_i^j C_n^m), \tag{61}$$

where $i' := n + 1 - i$, $\epsilon = 1$ for $O_q(N)$, $\epsilon = -1$ for $Sp_q(N)$ and $C = (C_j^i)$ is the corresponding matrix of the metric (see [5] or [6] for details). We shall essentially use the fact that $C_j^i = 0$ if $i \neq j'$.

Before we turn to the construction of the FODC, let us look for a moment at the "ordinary" first order calculus on the Lie groups $O(N)$ and $Sp(N)$. Then the matrix

$$\theta = S(\mathbf{u})\,\mathrm{d}\mathbf{u} = (\theta_{ij} \equiv \sum_k S(u_k^i)\,\mathrm{d}u_j^k)_{i,j=1,\dots,N} \tag{62}$$

satisfies the relation

$$\theta^t = -C^{-1}\theta C, \quad \text{i.e.,} \quad \theta_{ji} = -(C^{-1})^i_{i'}\theta_{i'j'}C_j^{j'} \quad \text{for } i, j = 1, \dots, N. \tag{63}$$

We briefly sketch the proof of this well-known fact. Indeed, differentiating the condition $\mathbf{u}^tC^{-1}\mathbf{u} = C^{-1}$, we obtain

$$\mathrm{d}\mathbf{u}^tC^{-1}\mathbf{u} + \mathbf{u}^tC^{-1}\,\mathrm{d}\mathbf{u} = 0. \tag{64}$$

From $C\mathbf{u}^tC^{-1}\mathbf{u} = I$ we get $C^{-1}S(\mathbf{u}) = \mathbf{u}^tC^{-1}$ and so $\mathbf{u}^tC^{-1}\,\mathrm{d}\mathbf{u} = C^{-1}\theta$. For the metric C of the Lie groups $O(N)$ and $Sp(N)$ we have $(C^{-1})^t = \epsilon C^{-1}$. Hence the relation $C^{-1}S(\mathbf{u}) = \mathbf{u}^tC^{-1}$ implies that $C^{-1}\mathbf{u} = S(\mathbf{u})^tC^{-1}$. Because functions and forms commute (!) for the classical differential calculus, we can write $\mathrm{d}\mathbf{u}^tC^{-1}\mathbf{u} = (S(\mathbf{u})\,\mathrm{d}\mathbf{u})^tC^{-1} = \theta^tC^{-1}$. Inserting these expressions into (64) we obtain (63).

For the construction of the left-covariant FODC we shall restrict ourselves to the quantum group $O_q(N)$. In the case of $Sp_q(N)$ one has to omit the elements X_{ii} supporting the secondary diagonal entries $u^i_{i'}$ in order to be in accordance with the ordinary calculus on the classical group $Sp(N)$. The remaining parts are verbatim the same.

Let us abbreviate $I := \{(i, j) : i' \le j, i, j = 1, \dots, N\}$. Then the elements u^i_j with $(i, j) \in I$ are precisely those entries of the matrix $\mathbf{u}$ that are below or on the secondary diagonal. Now we define

$$X_{ji} := l_i^{-j} l_j^{+j} Z_j \quad \text{and} \quad X_{ij} := l_{i'}^{+j'} l_{j'}^{+j'} Z_{j'} \qquad \text{for } j' \le i < j, \quad i, j = 1, \dots, n,$$

where Z_j and $Z_{j'}$ are group-like elements of the Hopf dual $\mathcal{O}(O_q(N))^\circ$. These functionals X_{ji}, X_{ij} separate the elements u^i_j such that $(i, j) \in I$ and $i \ne j$. In order to separate also the entries $u^i_{i'}$, $i' \le i$, we choose group-like elements Y_i, $i' \le i$, of $\mathcal{O}(O_q(N))^\circ$ such that

$$(Y_i - \varepsilon, u^r_s) = \delta_{rs}\delta_{ir} \quad \text{for } (r, s) \in I, \ \ i' \le i, \tag{65}$$

and put

$$X_{ii} := Y_i - \varepsilon \quad \text{for } i' \le i, \ \ i = 1, \dots, N.$$

Further, we suppose that

$$Z_j, Z_{j'} \in \mathrm{Lin}\,\{Y_i; i' \le i\} \quad \text{for } j' < j. \tag{66}$$

Then the vector space $\mathcal{T} = \mathrm{Lin}\{X_{rs}; (s, r) \in I\}$ is the quantum tangent space of a left-covariant FODC Γ on $\mathcal{O}(O_q(N))$. From the construction and the explicit form of the matrix R it is straightforward to check that $(X_{rs}, u^i_j) \ne 0$ if and only if $(r, s) = (j, i)$ for arbitrary indices $(s, r) \in I$ and $(i, j) \in I$. This implies that the FODC Γ has the dimension $N(N + 1)/2$ and that the elements $\theta_{ij} = \omega(u^i_j)$, $(i, j) \in I$, form a basis of the space of left-invariant one-forms ${}_{\mathrm{inv}}\Gamma$. These facts are in accordance with the ordinary first order calculus on the Lie group $O(N)$. Note that the FODC Γ just constructed depends on the group-like elements $Z_j, Z_{j'} j' < j$, and $Y_i, i' \le i$, of $\mathcal{O}(O_q(N))^\circ$ which can be freely chosen such that they satisfy the assumptions (65) and (66). These conditions can be easily fulfilled by taking monomials in the main diagonal L-functionals $l_i^{\pm i}$. We make all that more explicit by an example.

Example. $\mathcal{O}(O_q(5))$. Then the 15 generators of the quantum tangent space $\mathcal{T}$ are

$$\begin{aligned}
X_{15} &= l_5^{+1} l_1^{-1} Z_1, & X_{25} &= l_4^{+1} l_1^{-1} Z_1,\\
X_{35} &= l_3^{+1} l_1^{-1} Z_1, & X_{45} &= l_2^{+1} l_1^{-1} Z_1,\\
X_{24} &= l_4^{+2} l_2^{-2} Z_2, & X_{34} &= l_3^{+2} l_2^{-2} Z_2,\\
X_{51} &= l_1^{+5} l_5^{-5} Z_5, & X_{52} &= l_2^{+5} l_5^{-5} Z_5,\\
X_{53} &= l_3^{+5} l_5^{-5} Z_5, & X_{54} &= l_4^{+5} l_5^{-5} Z_5,\\
X_{42} &= l_2^{+4} l_4^{-4} Z_4, & X_{43} &= l_3^{+4} l_4^{-4} Z_4,
\end{aligned}$$

$$X_{33} = Y_3 - \varepsilon, \qquad X_{44} = Y_4 - \varepsilon, \qquad X_{55} = Y_5 - \varepsilon$$

and assumption (66) means that $Z_1, Z_2, Z_4, Z_5 \in \mathrm{Lin}\{Y_3, Y_4, Y_5\}$. □

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